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The field boundary of two line currents in a plasma at uniform pressure

By C. SOZOU†

Mathematics Department, Northampton College of Advanced Technology, London, E.C. 1

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A solution is obtained for the boundary of the magnetic field of two arbitrary line currents immersed in a plasma at uniform pressure. The linear dimensions of the boundary, which are determined for several sets of data, are proportional to the intensity of the line currents (assuming that their ratio is constant) and inversely proportional to the square root of the plasma pressure. The cusps over the boundary, which when the two line currents are equal and opposite are symmetrical with respect to the line currents, move towards the weaker current until they eventually disappear on the line joining the two currents. When the distance between the currents is such that when the cusps disappear the two line currents lie in the same cavity, some plasma gets trapped inside the cavity. When the cavities of two line currents of the same sign touch we get an abrupt instability; the plasma is suddenly pushed out by the surface currents and this results in the formation of a larger field cavity.

1. Introduction

The cavity in which sources of magnetic field are confined by a plasma has been studied by several authors in an attempt to estimate theoretically the shape and size of the cavity carved out of the solar plasma by the geomagnetic field. The problem of determining the cavity in which a three-dimensional dipole field is confined by a streaming plasma is intractable and has been studied only by approximate methods (see Beard 1960; Midgley & Davis 1963; Mead & Beard 1964). Midgley & Davis (1962) and Slutz (1962) used numerical methods to estimate the cavity of a three-dimensional dipole field in a uniform plasma pressure.

Fortunately, exact solutions can be obtained for two-dimensional models by the use of conformal mapping. Thus the problem of a cold plasma streaming past a two-dimensional dipole has been solved independently by Dungey (1961), Hurley (1961) and Zhigulev & Romishevskii (1959).

The problem of a two-dimensional dipole field under uniform plasma pressure has been examined by the present author (1964). In this note we extend this case and determine the cavity in which the magnetic field of two arbitrary line currents is confined by an external plasma at a uniform pressure. We assume an equilibrium state such that at the boundary of the cavity there is a thin current

† Now at the University of Sheffield.

sheath with the plasma outside the cavity. We also assume that the location and intensity of the line currents is such that both of them are in one and the same cavity.

We obtain an exact solution, but when the location and intensity of the line currents is given we have to solve numerically a pair of equations for the determination of two constants in order that we may be able to specify the boundary. Thus it is simpler to determine a boundary that corresponds to a given ratio of the intensity of the line currents and then specify the location of the line currents.

2. Equations of the problem

Let the magnetic field be $\mathbf{B} = (B_x, B_y, 0)$. This must satisfy the following conditions:

$$\left.\begin{aligned} \nabla . \mathbf{B} &= 0 \text{ everywhere} \\ \nabla \times \mathbf{B} &= 0 \text{ everywhere inside the cavity except at the} \\ &\quad\ \text{line currents.} \end{aligned}\right\} \tag{1}$$

At the boundary

$$|\mathbf{B}|^2 = B_x^2 + B_y^2 = p, \tag{2}$$

where $p/8\pi$ is the uniform plasma pressure. (1) is satisfied if

$$\bar{B} = B_x - iB_y = \bar{B}(z),$$

where $z = x + iy$ and a bar denotes complex conjugate.

Since at the boundary the magnetic field is tangential we have

$$\bar{B}\,dz = B_x dx + B_y dy = \mathbf{B}.d\mathbf{s} = \pm |\mathbf{B}|ds = \pm p^{\frac{1}{2}} ds \tag{3}$$

using (2), where ds is an element of arc of the boundary. In (3) the positive sign holds when $\mathbf{B}$ and $d\mathbf{s}$ are in the same direction and the negative sign holds when $\mathbf{B}$ and $d\mathbf{s}$ are in opposite directions.

Let

$$\bar{B} = d\phi/dz, \tag{4}$$

and let the two line currents be I_1 and I_2 situated at a_1 and a_2, respectively. If we assume that very near the line currents the lines of force are not affected substantially by the plasma pressure, we require

$$\left.\begin{aligned} d\phi/dz = \bar{B}(z) &\sim -2iI_1/(z-a_1) \quad \text{when} \quad z \to a_1, \\ &\sim -2iI_2/(z-a_2) \quad \text{when} \quad z \to a_2. \end{aligned}\right\} \tag{5}$$

Let us assume (Riemann's mapping theorem) that there is a transformation $w(z)$ [and its inverse $z(w)$] that transforms the unknown domain in the z-plane into a unit circle in the w-plane so that the origins in the two domains correspond. Let also the points a_1 and a_2 in the z-plane correspond to points A_1 and A_2, respectively, in the w-plane, that is

$$\left.\begin{aligned} w(a_1) &= A_1 \quad \text{or} \quad z(A_1) = a_1, \\ w(a_2) &= A_2 \quad \text{or} \quad z(A_2) = a_2. \end{aligned}\right\} \tag{6}$$

If we assume a_1 and a_2 to be real, the symmetry of the problem requires that A_1 and A_2 are also real. If ϕ is expressed in terms of w we have

$$\phi(z) = g(w). \tag{7}$$

From (5), (6) and (7) we get

$$\left.\begin{aligned}\frac{dg}{dw} = \frac{d\phi}{dz}\frac{dz}{dw} &\sim -\frac{2iI_1}{w-A_1} \quad \text{as} \quad w \to A_1,\\ &\sim -\frac{2iI_2}{w-A_2} \quad \text{as} \quad w \to A_2,\end{aligned}\right\} \tag{8}$$

that is, the singularities of $d\phi/dz$ in the z-plane become singularities of dg/dw at the corresponding points in the w-plane. Equations (3), (4) and (7) show that dg is real on the circle $|w| = 1$. This, and the fact that dg/dw has two simple poles at A_1 and A_2 inside the unit circle, requires that

$$\frac{dg}{dw} = -2i\left(\frac{I_1}{w-A_1} - \frac{I_1A_1}{A_1w-1} + \frac{I_2}{w-A_2} - \frac{A_2I_2}{A_2w-1}\right). \tag{9}$$

Since $d\phi/dz = (dg/dw)\,(dw/dz)$, from (3), (4) and (9) we find that at the boundary ($w = e^{i\theta}$)

$$\left|\frac{dz}{dw}\right| = \frac{2}{p^{\frac{1}{2}}}\left|\frac{I_1(1-A_1^2)}{1+A_1^2-2A_1\cos\theta} + \frac{I_2(1-A_2^2)}{1+A_2^2-2A_2\cos\theta}\right|. \tag{10}$$

Since $w(z)$ transforms the domain in the z-plane conformally into a circle in the w-plane, dw/dz (and dz/dw) is regular in the domain, that is, it has no zeros or singularities. If the boundary has cusps the transformation is not conformal there and dz/dw is not regular. This may happen if the two line currents are flowing in opposite directions. In this case we may get two cusps symmetrical with respect to the line joining the two line currents. All these conditions and equation (10) are satisfied by

$$\frac{p^{\frac{1}{2}}}{2I}\frac{dz}{dw} = \frac{(\Lambda - Xw)(\bar{\Lambda}-\bar{X}w)}{(A_1w-1)^2(A_2w-1)^2}, \tag{11}$$

where $I = I_1$ (assumed different from zero) and

$$\left.\begin{aligned}2\Lambda &= \{(1-A_1^2)(1+A_2)^2 + C(1-A_2^2)(1+A_1)^2\}^{\frac{1}{2}}\\ &\qquad + \{(1-A_1^2)(1-A_2)^2 + C(1-A_1)^2(1-A_2^2)\}^{\frac{1}{2}}\\ &= (1+A_1)(1+A_2)\left(\frac{1-A_1}{1+A_1} + C\frac{1-A_2}{1+A_2}\right)^{\frac{1}{2}}\\ &\qquad + (1-A_1)(1-A_2)\left(\frac{1+A_1}{1-A_1} + C\frac{1+A_2}{1-A_2}\right)^{\frac{1}{2}},\end{aligned}\right\} \tag{12}$$

$$\begin{aligned}2X &= (1+A_1)(1+A_2)\left(\frac{1-A_1}{1+A_1} + C\frac{1-A_2}{1+A_2}\right)^{\frac{1}{2}}\\ &\qquad - (1-A_1)(1-A_2)\left(\frac{1+A_1}{1-A_1} + C\frac{1+A_2}{1-A_2}\right)^{\frac{1}{2}},\end{aligned} \tag{13}$$

and $C = I_2/I_1$. Equations (11), (12) and (13) show that dz/dw may vanish at conjugate points on the boundary ($|w| = 1$) if I_1 and I_2 are of opposite sign. In this case we get two cusps at the boundary and the transformation is not conformal there.

Integrating (11) and arranging the constant of integration so that the origins in the two planes correspond we get the required transformation

$$\frac{p^{\frac{1}{2}}z}{2I} = \left\{\frac{(A_1\Lambda - X)(A_1\bar{\Lambda} - \bar{X})}{1-A_1 w} + \frac{(A_2\Lambda - X)(A_2\bar{\Lambda} - \bar{X})}{1-A_2 w}\right\}\frac{w}{(A_2-A_1)^2} - F\log\frac{1-A_1 w}{1-A_2 w}, \tag{14}$$

where

$$F = \{(X - A_1\Lambda)(\bar{X} - A_2\bar{\Lambda}) + (\bar{X} - A_1\bar{\Lambda})(X - A_2\Lambda)\}/(A_2 - A_1)^3. \tag{15}$$

If we choose a_1 negative and a_2 positive, the corresponding points A_1 and A_2 will be negative and positive, respectively. If $a_1 = -a_2$, equations (6) and (14) require that

$$-\frac{p^{\frac{1}{2}}a_2}{2I} = \left[\frac{(X - A_1\Lambda)(\bar{X} - A_1\bar{\Lambda})}{1-A_1^2} + \frac{(X - A_2\Lambda)(\bar{X} - A_2\bar{\Lambda})}{1-A_1A_2}\right]\frac{A_1}{(A_2-A_1)^2} - F\log\left(\frac{1-A_1^2}{1-A_1A_2}\right), \tag{16}$$

$$\frac{p^{\frac{1}{2}}a_2}{2I} = \left[\frac{(X - A_1\Lambda)(\bar{X} - A_1\bar{\Lambda})}{1-A_1A_2} + \frac{(X - A_2\Lambda)(\bar{X} - A_2\bar{\Lambda})}{1-A_2^2}\right]\frac{A_2}{(A_2-A_1)^2} - F\log\left(\frac{1-A_1A_2}{1-A_2^2}\right). \tag{17}$$

When the two line currents are of the same intensity $A_1 = -A_2$ and (17) becomes identical with (16). In this case (16) becomes

$$p^{\frac{1}{2}}a_2/I = A_2[2 + (A_2^{-2} - A_2^2)\log\{(1+A_2^2)/(1-A_2^2)\}], \tag{18}$$

when the two currents are flowing in the same direction, and

$$p^{\frac{1}{2}}a_2/I = 2 + (A_2^{-1} - A_2)^2\log\{(1-A_2^2)/(1+A_2^2)\}, \tag{19}$$

when they are flowing in opposite directions.

3. Results and discussion

For a given set of data we have to solve equations (16) and (17) for A_1, A_2 and then use (14) to determine the boundary. Thus it would be simpler to prescribe A_1, A_2 and the ratio of the intensity of the line currents and use (14) to determine the boundary and the position of the line currents. It follows from (14) that the linear dimensions of the boundary are proportional to $I/p^{\frac{1}{2}}$.

When the two line currents are of the same intensity we obtain the boundary by prescribing A_1 and A_2 in (14). The distance between the line currents is obtained from (18) when the line currents are flowing in the same direction, or from (19) when the line currents are flowing in opposite directions. The results are shown in table 1 and figure 1.

The right-hand side of (19) is a monotonic function of A_2 increasing from zero when $A_2 = 0$, to 2 when $A_2 = 1$. When A_2 is small we get a dipole-like boundary (Sozou 1964). When A_2 increases, that is when the distance between the line currents increases, the cavity expands (figure 1) and the cusps become deeper.

As $A_2 \to 1$ the boundary tends to become two circles with the cusps going down to the origin. In the limit ($A_2 = 1$) we get two circles touching at the origin, and since our domain is not simple the transformation breaks down.

Equation (18) has two roots, that is two values of A_2, when the distance between the line currents is greater than the sum of the radii of the circular cavities in which each current is separately confined by the plasma. This implies that in

Curve	A_1	A_2	$2p^{\frac{1}{2}}a_2/I_1$	I_2/I_1
A	-0.7795	0.7795	5.4004	1
B	-0.8319	0.8319	5.4620	1
C	-0.9	0.9	5.3226	1
D	-0.5119	0.5119	4	1
E	-0.25	0.25	1.9974	1
A	-0.9	0.9	3.7990	-1
B	-0.75	0.75	3.1336	-1
C	-0.50	0.50	1.7012	-1
D	-0.25	0.25	0.4798	-1

TABLE 1

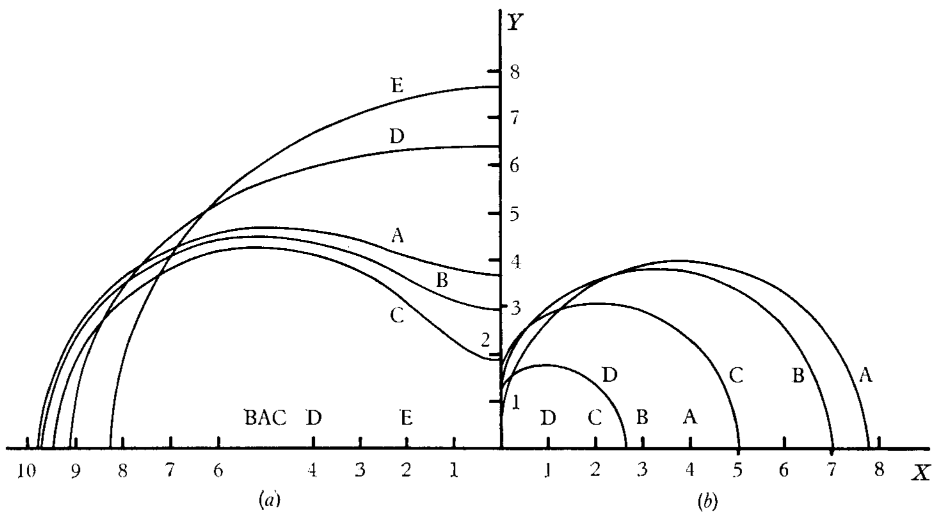

FIGURE 1. (*a*) Currents of the same sign. (*b*) Currents of opposite sign. Quarter cross-section of the cavity of two line currents of equal intensity equidistant from the origin, in a plasma at uniform pressure. Letters show the boundary and the position of the currents on the real axis. $X = 2p^{\frac{1}{2}}x/I$, $Y = 2p^{\frac{1}{2}}y/I$.

this case we may get two different cavities for the same set of data. One of the cavities, however, is unstable. Physically this can be explained as follows: At the boundary the magnetic field is tangential and thus the surface current is proportional to the magnetic field. Since the magnitude of the magnetic field at the boundary is constant and since the sum of the surface currents is constant–equal and opposite to the inducing currents inside the cavity—the length of the boundary is constant. Thus, when we increase the distance between the line currents the cavity expands along the axis of the currents and contracts perpendicular to it and eventually a waist develops over the boundary. When the position A (figure 1) is reached (corresponding to $A_2 = 0{\cdot}7795$), the length of the cavity along the line of the currents is a maximum. If we increase the distance between the line currents further, the cavity contracts in all directions until the position B (corresponding to $A_2 = 0{\cdot}8319$) is reached. If we now try to separate the line currents further, the magnetic field at the centre of the waist becomes too weak to stand the plasma pressure and the one cavity domain breaks down abruptly.

Curve	A_1	A_2	$2p^{\frac{1}{2}}a_2/I_1$	I_2/I_1
A	$-0{\cdot}2255$	$0{\cdot}2454$	1	$0{\cdot}1$
B	$-0{\cdot}2864$	$0{\cdot}3679$	1	$-0{\cdot}1$
C	$-0{\cdot}3153$	$0{\cdot}4839$	1	$-0{\cdot}135$
D	$-0{\cdot}3338$	$0{\cdot}6486$	1	$-0{\cdot}145$
E	$-0{\cdot}3620$	$0{\cdot}5010$	1	$-0{\cdot}50$

TABLE 2

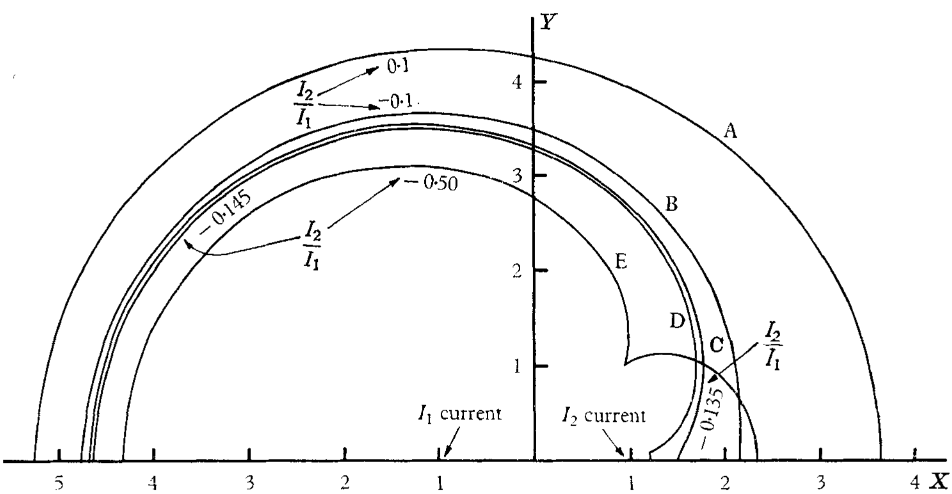

FIGURE 2. Half the cross-section of the cavity of two line currents in a plasma at uniform pressure. $X = 2p^{\frac{1}{2}}x/I_1$, $Y = 2p^{\frac{1}{2}}y/I_1$.

If, when the position B (figure 1) is reached, we reduce the distance between the line currents, we either get a position the reverse of that obtained when the distance between the line currents was increasing (this corresponds to values of $A_2 < 0{\cdot}8319$) or the cavity contracts in all directions (this corresponds to $0{\cdot}8319 < A_2 \leqslant 1$), and the waist increases until the limiting case ($A_2 = 1$) when the transformation breaks down and we get two circles touching at the origin. Curve C (figure 1) is a curve obtained from the second alternative. We think that the second alternative may represent an unstable situation.

The fact that there is only one solution of (18) when the distance between the line currents is less than the sum of the radii of the circular cavities in which each current is separately confined by the plasma means that when the circular boundaries of two line currents flowing in the same direction are brought to contact we get an abrupt instability (explosion) and a larger cavity enclosing both currents. This happens because of the overlapping of the surface currents at the point of contact and the resulting sudden increase there of the Lorentz force by a factor of 2. The Lorentz force pushes the plasma quickly outwards. This sets the surface currents in motion outwards from the cavity and thus we get a larger cavity (curve D of figure 1) containing both currents.

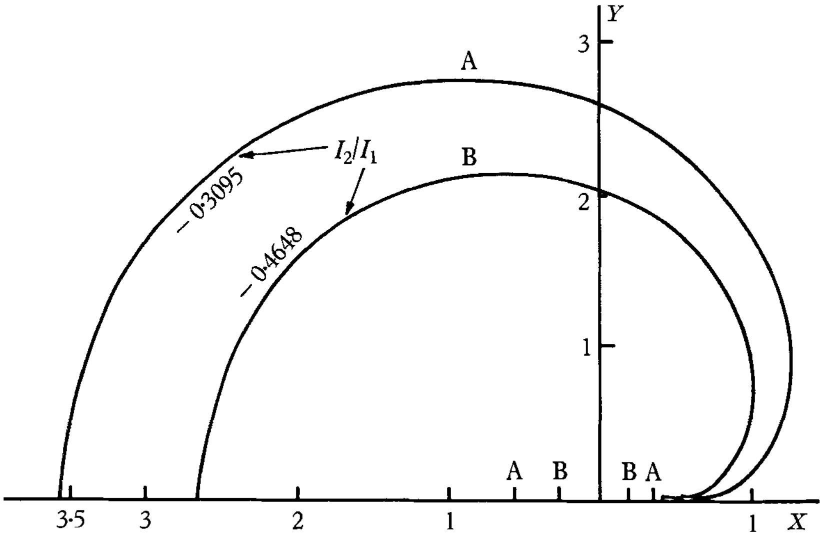

FIGURE 3. Half the cross-section of the cavity of two line currents, when the boundary touches the axis of the currents, in a plasma at uniform pressure. Letters show the boundary and the positions of the currents on the real axis. $X = 2p^{\frac{1}{2}}x/I_1$, $Y = 2p^{\frac{1}{2}}y/I_1$.

Curve	A_1	A_2	$2p^{\frac{1}{2}}a_1/I_1$	$2p^{\frac{1}{2}}a_2/I_1$	I_2/I_1
A	$-0{\cdot}30$	$0{\cdot}30$	$-0{\cdot}5686$	$0{\cdot}3201$	$-0{\cdot}3095$
B	$-0{\cdot}20$	$0{\cdot}20$	$-0{\cdot}2710$	$0{\cdot}1854$	$-0{\cdot}4648$

TABLE 3

Qualitatively, similar results are of course expected when the two line currents are of different intensities. Figure 2 shows boundaries produced by line currents of different intensities. The corresponding data (A_1 and A_2 were obtained from equations (16) and (17)) are shown in table 2.

When the two line currents are flowing in opposite directions the cusps move towards the weaker current and eventually disappear on the axis of the currents. When the distance between the line currents is smaller than the sum of the radii of the circular cavities in which each current is separately confined by the plasma, the angle between the axis of the currents and the cusps gets smaller as the distance between the currents is reduced or as the weak current gets weaker (figure 2). Thus eventually the cusps become parallel to the axis of the currents and maxima of the boundary. In the limiting case the boundary closes, touching the axis of the currents behind the cusps, and thus it encloses some plasma within the cavity. This geometrical configuration is shown in figure 3. Table 3 shows the corresponding data. The surface currents at the points of contact (figure 3) are flowing in the same direction. Thus in this case we get again an abrupt instability similar to the one considered before. The boundary becomes a continuous curve but with some plasma trapped inside the cavity. The surface currents that are also trapped in the cavity are equal and opposite and cancel out. This mechanism of trapping plasma inside a field cavity may be of some importance in astrophysics.

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